Directed Graphs

Jayati Kaushik

St. Joseph's University, Bengaluru

Foundations of Data Science
BDA2121

Directed Graphs

Notation and Definitions

Definiton 1

A *directed graph* (digraph) is a 4-tuple $G = (V, R, \alpha, \omega)$ with the following properties:

1. $V \neq \varnothing$ is the vertex set.
2. R the set of edges
3. $V \cap R = \varnothing$
4. $\alpha : R \to V$ and $\omega : R \to V$ are maps where $\alpha(r)$ is the starting vertex of the edge $r \in R$ and $\omega(r)$ is the ending vertex of r.

A graph G is *finite* if both V and R are finite.

Definition 2

Let G be a graph. We call $r \in R(G)$ a loop, if $\alpha(r) = \omega(r)$.
We call r, r' *parallel*(*anti-parallel*) if $\alpha(r) = \alpha(r')$ $(\alpha(r) = \omega(r'))$ and $\omega(r) = \omega(r')(\omega(r) = \alpha(r'))$
We call G *simple* if it does not contain any parallels or loops. In this case, we can identify $r \in R(G)$ with the pair $(\alpha(r), \omega(r))$.

Definition 3

We call $v \in V(G)$ and $r \in R(G)$ *incident* if $v \in \{\alpha(r), \omega(r)\}$
Two vertices $v \neq u$ are *adjecent*, if there is an arc r which is incident to u and v.

Definition 4

We call $G = (V, R, \alpha, \omega)$ and $G' = (V', R', \alpha', \omega')$ *isomorphic* and write $G \cong G'$ if there are bijections $\sigma : V \to V'$, $\tau : R \to R'$ such that for all $r \in R$ we

1. $\alpha'(\tau(r)) = \sigma(\alpha(r))$
2. $\omega'(\tau(r)) = \sigma(\omega(r))$

Subgraphs

Definition 6

We call $G' = (V', R', \alpha', \omega')$ a *subgraph* of $G = (V, R, \alpha, \omega)$ if:

1. $V' \subseteq V$, $R' \subseteq R$
2. $\alpha_{|_{R'}} = \alpha'$, $\omega_{|_{R'}} = \omega'$

We write $G' \sqsubseteq G$

The relation "$\sqsubseteq$" satisfies:

1. $G \sqsubseteq G$ (reflexivity)
2. $G \sqsubseteq G', G' \sqsubseteq G \implies G = G'$ (antisymmetry)
3. $G \sqsubseteq G', G' \sqsubseteq G'' \implies G \sqsubseteq G''$ (transitivity)

Defintion 7

For $S \subseteq V(G)$ we define $G[S]$ to be the subgraph with the vertex S and arc set $\{r \in R : \alpha(r) \in S \text{ and } \omega(r) \in S\}$. We call $G[S]$ the *induced subgraph* of S in G. Further for $A \subseteq R$ let $G_A := (V, A, \alpha_{|_A}, \omega_{|_A})$

Undirected Graphs

Definition 8

An *undirected graph* is a triple $G = (V, E, \gamma)$ consisting of a nonempty set V, a set E with $V \cap E = \emptyset$ and a mapping

$$\gamma : E \to \{X : X \subseteq V, 1 \leq | X | \leq 2\}$$

Incidence and adjacency are defined analogously to the directed case.

Let $n := \mid V(G) \mid$ and $m := \mid R(G) \mid$

Adjacency Matrix Representation

The adjacency matrix $A(G)$ is the $n \times n$ matrix with

$$a_{ij} = \mid \{r \in R : \alpha(r) = v_i and \omega(r) = v_j\} \mid$$

$\delta^+(v) := \{r \in R : \alpha(r) = v\}$ $\delta^-(v) := \{r \in R : \omega(r) = v\}$
$\delta(v) := \{e \in E : v \in \gamma(e)\}$

Adjacency List Representation

We store n, m and an array Adj[v] of size n
Adj[v] is a list of all the arc in $\delta^+(v)$